Lecture 3

Learning outcomes:

- *Numerical Calculations with NumPy:*
  - *Eigenvalues and eigenvectors*

Eigenvalues and Eigenvectors

- ❑ Eigenvectors are the directions that remain unchanged during a transformation, even if they get longer or shorter.
- ❑ Eigenvalues are the numbers that indicate how much something stretches or shrinks during that transformation.
- ✓ These ideas are important in many areas of math and engineering, including studying how systems stay stable and understanding quantum physics. They’re also important in fields like machine learning, where they help simplify complex data by reducing its dimensions.
- ➢ Eigenvalues and Eigenvectors are the scalar and vector quantities associated with matrices used for linear transformations. The vector that only changes by a scalar factor after applying a transformation is called an eigenvector, and the scalar value attached to the eigenvector is called the eigenvalue.

Eigenvalues Definition

- Eigenvalues are the scalar values associated with the eigenvectors in linear transformation. The word ‘Eigen’ is of German Origin which means ‘characteristic’.
- Hence, these characteristic values indicate the factor by which eigenvectors are stretched in their direction. It doesn’t involve the change in the direction of the vector except when the eigenvalue is negative. When the eigenvalue is negative the direction is just reversed.
- The equation for eigenvalue is given by:

$Av = \lambda v$

Where,

✓A is the matrix,

✓v is associated eigenvector, and

✓λ is scalar eigenvalue.

What are Eigenvectors?

- Eigenvectors for square matrices are defined as non-zero vector values which when multiplied by the square matrices give the scaler multiple of the vector, i.e. we define an eigenvector for matrix A to be "v" if it specifies the condition, $Av = \lambda v$.

- The scaler multiple λ in the above case is called the eigenvalue of the square matrix. We always have to find the eigenvalues of the square matrix first before finding the eigenvectors of the matrix.

- For any square matrix, A of order $n \times n$ the eigenvector is the column matrix of order $n \times 1$. If we find the eigenvector of the matrix A by, $Av = \lambda v$, "v" in this is called the right eigenvector of the matrix A and is always multiplied to the right-hand side as matrix multiplication is not commutative in nature. In general, when we find the eigenvector it is always the right eigenvector.

- We can also find the left eigenvector of the square matrix A by using the relation, $vA = v\lambda$

- Here, v is the left eigenvector and is always multiplied to the left-hand side. If matrix A is of order $n \times n$ then v is a column matrix of order $1 \times n$.

Eigenvector Equation

- The Eigenvector equation is the equation that is used to find the eigenvector of any square matrix. The eigenvector equation is,

$Av = \lambda v$

Where,

✓A is the given square matrix,

✓v is the eigenvector of matrix A, and

✓λ is any scaler multiple.

What are Eigenvalues and Eigenvectors?

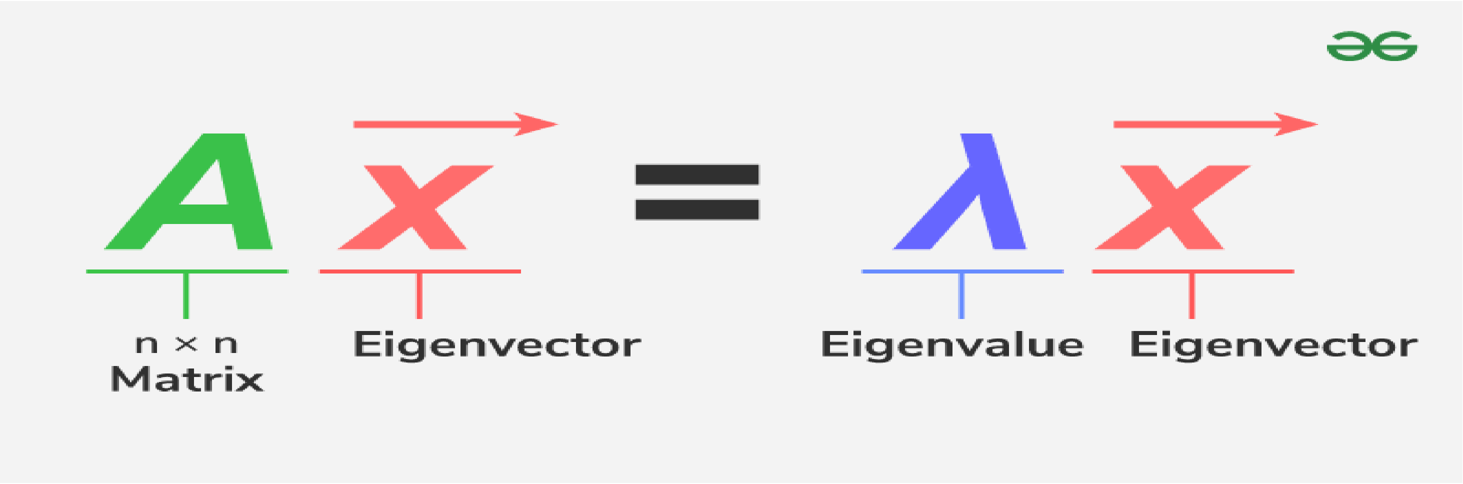
- If A is a square matrix of order n × n then we can easily find the eigenvector of the square matrix by following the method discussed below,
- We know that the eigenvector is given using the equation $Av = \lambda v$, for the identity matrix of order same as the order of A i.e. n × n we use the following equation,

$$(A-\lambda I)v = 0$$

✓Solving the above equation we get various values of λ as λ1, λ2, …, λn these values are called the eigenvalues and we get individual eigenvectors related to each eigenvalue.

✓Simplifying the above equation we get v which is a column matrix of order n × 1 and v is written as,

$$V = \begin{bmatrix} v1 \\ v2 \\ v3 \\ . \\ . \\ vn \end{bmatrix}$$



How to Find an Eigenvector?

- The eigenvector of the following square matrix can be easily calculated using the steps below,

- ➢Step 1: Find the eigenvalues of the matrix A, using the equation det $|(A – \lambda I|$ =0, where “I” is the identity matrix of order similar to matrix A
- ➢Step 2: The value obtained in Step 2 are named as, $\lambda 1$, $\lambda 2$, $\lambda 3$….
- ➢Step 3: Find the eigenvector (X) associated with the eigenvalue $\lambda 1$ using the equation, $(A – \lambda 1 I) X = 0$
- ➢Step 4: Repeat step 3 to find the eigenvector associated with other remaining eigenvalues $\lambda 2$, $\lambda 3$….

✓Following these steps gives the eigenvector related to the given square matrix.

Types of Eigenvector

❖The eigenvectors calculated for the square matrix are of two types which are,

➢Right Eigenvector

➢Left Eigenvector

- Right Eigenvector

The eigenvector which is multiplied by the given square matrix from the right-hand side is called the right eigenvector. It is calculated by using the following equation,

$$AV_R = \lambda V_R$$

Where,

A *is given square matrix of order* $n \times n$,

λ *is one of the eigenvalues, and*

V_R *is the column vector matrix*

The value of $\boldsymbol{V_R}$ is,

$$\boldsymbol{V_R} = \begin{bmatrix} v1 \\ v2 \\ v3 \\ . \\ . \\ vn \end{bmatrix}$$

- Left Eigenvector

The eigenvector which is multiplied by the given square matrix from the left-hand side is called the left eigenvector. It is calculated by using the following equation,

$$V_L A = V_L \lambda$$

Where,

A *is given square matrix of order n×n,*

λ *is one of the eigenvalues, and*

V_L *is the row vector matrix.*

The value of VL is,

$$V_L = [V_1, V_2, V_3, \ldots, V_n]$$

❑ Eigenvectors of a Square Matrix

We can easily find the eigenvector of square matrices of order n × n. Now, let’s find the following square matrices:

- ➢Eigenvectors of a 2 × 2 matrix
- ➢Eigenvectors of a 3 × 3 matrix.

▪ Eigenvector of a 2 × 2 matrix

The Eigenvector of the 2 × 2 matrix can be calculated using the above mention steps. An example of the same is,

Example: Find the eigenvalues and the eigenvector for the matrix A =

1	2
5	4

Solution:

- If eigenvalues are represented using λ and the eigenvector is represented as v =

a
b

- Then the eigenvector is calculated by using the equation,

$|A - \lambda I| = 0$

1	**2**
5	4

$- \lambda$

1	**0**
0	1

$=$

0	**0**
0	0

1-λ	**2**
5	4-λ

$= 0$

$(1-\lambda)(4-\lambda) - 2.5 = 0$

$\Rightarrow 4 - \lambda - 4\lambda + \lambda^2 - 10 = 0$

$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$

$\Rightarrow \lambda^2 - 6\lambda + \lambda - 6 = 0$

$\Rightarrow \lambda(\lambda-6) + 1(\lambda-6) = 0$

$\Rightarrow (\lambda-6)(\lambda+1) = 0$

Factorizing: $\lambda = 6$ *and* $\lambda = -1$

✓ *Thus, the eigenvalues are 6, and -1. Then the respective eigenvectors are,*

For $\boldsymbol{\lambda = 6}$

$(A-\lambda I)v = 0$

$$\Rightarrow \begin{bmatrix} 1-6 & 2 \\ 5 & 4-6 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -5 & 2 \\ 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$\Rightarrow$ -5a + 2b = 0

$\Rightarrow$ 5a – 2b = 0

Simplifying the above equation we get,

5a=2b

The required eigenvector is, $\begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

For λ = -1

$(A-\lambda I)v = 0$

$$\Rightarrow \begin{bmatrix} 1-(-1) & 2 \\ 5 & 4-(-1) \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$\Rightarrow 2a + 2b = 0$

$\Rightarrow 5a + 5b = 0$

simplifying the above equation we get,

a = -b

The required eigenvector is, $\begin{bmatrix} a \\ b \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then the eigenvectors of the given 2 × 2 matrix are

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- Eigenvector of a 3 × 3 Matrix

The Eigenvector of the 3 × 3 matrix can be calculated using the above mention steps. An example of the same is,

➢ Example: Find the eigenvalues and the eigenvector for the matrix A =

2	**2**	**2**
2	2	2
2	2	2

Solution:

- If eigenvalues are represented using λ and the eigenvector is represented as v =

a
b
c

- Then the eigenvector is calculated by using the equation,

$|A- \lambda I| = 0$

2	**2**	**2**
2	2	2
2	2	2

-λ

1	**0**	**0**
0	1	0
0	0	1

=

0	**0**	**0**
0	0	0
0	2	0

2-λ	**2**	**2**
2	2-λ	2
2	2	2-λ

$= 0$

Simplifying the above determinant we get

$$\Rightarrow (2-\lambda)(\lambda^2) + 2\lambda^2 + 2\lambda^2 = 0$$

$$\Rightarrow (-\lambda^3) + 6\lambda^2 = 0$$

$$\Rightarrow \lambda^2(6 - \lambda) = 0$$

$$\Rightarrow \lambda = 0, \lambda = 6$$

For λ = 0

$$(A - \lambda I)\, v = 0$$

$$\Rightarrow \begin{bmatrix} 2-0 & 2 & 2 \\ 2 & 2-0 & 2 \\ 2 & 2 & 2-0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Simplifying the above equation we get

$2a + 2b + 2c = 0$

$\Rightarrow 2(a+b+c) = 0$

$\Rightarrow a+b+c = 0$

Let $b = k_1$ *and* $c = k_2$

$a + k_1 + k_2 = 0$

$a = -(k_1 + k_2)$

Thus, the eigenvector is,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -(k_1 + k_2) \\ k_1 \\ k_2 \end{bmatrix}$$

taking $k_1 = 1$ and $k_2 = 0$

the eigenvector is,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

taking $k_1 = 0$ and $k_2 = 1$

the eigenvector is,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$For\ \lambda = 6$

$(A - \lambda I)\, v = 0$

$\Rightarrow$

2-6	**2**	**2**
2	2-6	2
2	2	2-6

.

a
b
c

$= 0$

$\Rightarrow$

-4	**2**	**2**
2	-4	2
2	2	-4

.

a
b
c

$=0$

Simplifying the above equation we get,

$-4a + 2b + 2c = 0$

$\Rightarrow 2(-2a + b + c) = 0$

$\Rightarrow -2a = -(b + c)$

$\Rightarrow 2a = b + c$

Let $b = k_1$ *and* $c = k_2$, *and taking* $k_1 = k_2 = 1$,

we get

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the eigenvector is,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Applications of Eigen Values in Engineering

❑Some of the common applications of eigen values are:

➢Linear Algebra

✓Diagonalization: Eigenvalues are used to diagonalize matrices, simplifying computations and solving linear systems more efficiently.

✓Matrix Exponentiation: Eigenvalues play a crucial role in computing the exponentiation of a matrix.

➢Quantum Mechanics

✓Schrödinger Equation: Eigenvalues of the Hamiltonian operator correspond to the energy levels of quantum systems, providing information about possible states.

✓Vibrations and Structural Analysis: Mechanical Vibrations: Eigenvalues represent the natural frequencies of vibrational systems. In structural analysis, they help understand the stability and behavior of structures.

- Statistics
  - ✓ Covariance Matrix: In multivariate statistics, eigenvalues are used in the analysis of covariance matrices, providing information about the spread and orientation of data.

- Computer Graphics
  - ✓ Principal Component Analysis (PCA): Eigenvalues are used in PCA to find the principal components of a dataset, reducing dimensionality while retaining essential information.

- Control Systems
  - ✓ System Stability: Eigenvalues of the system matrix are critical in determining the stability of a control system. Stability analysis helps ensure that the system response is bounded.

Numerical Calculations of Eigenvalues/Eigenvectors in python

- Unsurprisingly, there is a function to calculate eigenvalues and eigenvectors in python!

- For most cases, we can use the np.linalg.eig function - Linear algebra (linalg).
- If we only wanted the eigenvalues, np.linalg.eigvals will just calculate those.

✓Syntax: numpy.linalg.eig()

✓Parameter: An square array.

✓Return: It will return two values first is eigenvalues and second is eigenvectors.

Example: Find the eigenvalues and the eigenvector for the matrix 2 × 2

```
import numpy as np
x = np.array([[1,2],
              [5,4]])

eigvals, eigvecs = np.linalg.eig(x)
print(eigvals)
print(eigvecs)
```

✓**Output**

```
[-1.  6.]
[[-0.70710678 -0.37139068]
 [ 0.70710678 -0.92847669]]
```

Example: Find the eigenvalues and the eigenvector for the matrix 3 × 3

```
import numpy as npy = np.array([[2,2,2],
                                [2,2,2],
                                [2,2,2]])
eigvals, eigvecs = np.linalg.eig(y)

print(eigvals)
print(eigvecs)
```

✓**Output**

```
[ 6.00000000e+00  0.00000000e+00 -3.03938812e-16]
[[ 5.77350269e-01 -1.16872145e-16 -8.11234711e-01]
 [ 5.77350269e-01 -7.07106781e-01  3.25469472e-01]
 [ 5.77350269e-01  7.07106781e-01  4.85765238e-01]]
```

Example: real roots

```
import numpy as np

A = np.array([[6,10,6],
              [0,8,12],
              [0,0,2]])

# Fill this in!
eigvals, eigvecs = np.linalg.eig(A)

# Notice that we can use variables in a print!
# f'something {var}' means sub in the var in the string
print(f'Eigenvalues = {eigvals}')
print(f'Eigenvectors: \n{eigvecs}')
```

✓**Output**:

Eigenvalues = [6. 8. 2.]

Eigenvectors:

```
[[ 1.          0.98058068  0.84270097]
 [ 0.          0.19611614 -0.48154341]
 [ 0.          0.          0.24077171]]
```

Example: Complex roots

```
A = np.array([[1,2],
              [-2,1]])

eigvals, eigvecs = np.linalg.eig(A)

print(f'Eigenvalues = {eigvals}')
print(f'Eigenvectors: \n{eigvecs}')
```

✓**Output:**

```
Eigenvalues = [1.+2.j 1.-2.j]
Eigenvectors:
[[0.        -0.70710678j 0.        +0.70710678j]
 [0.70710678+0.j         0.70710678-0.j        ]]
```