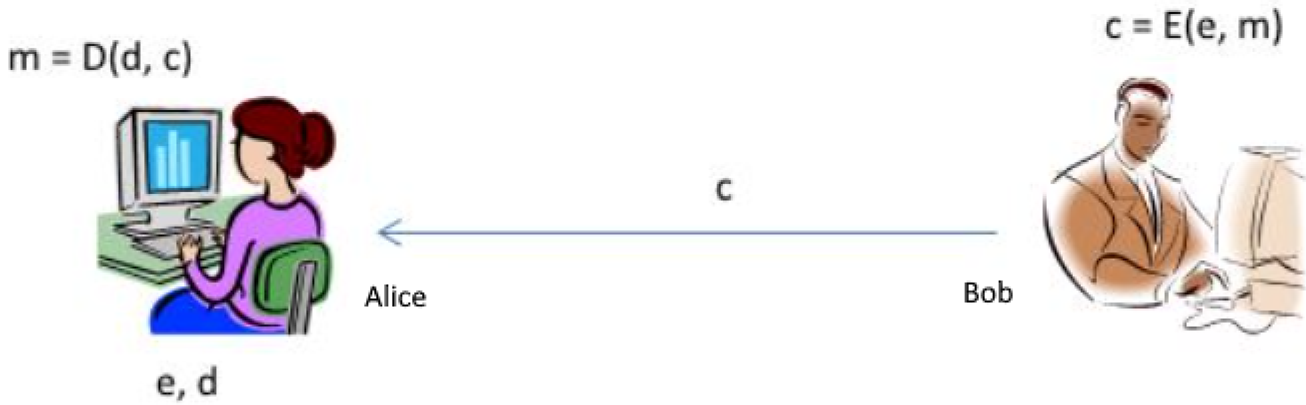
The Basic Model of Public Key Systems



1. Alice generates a pair of keys: **e** (public key) and **d** (private key). In this context **e** means (**e**ncryption key) and **d** (**d**ecryption key).

2. She keeps **d** secret, but makes **e** public.

3. If Bob wants to send a message to Alice, he uses Alice's public key **e**.

4. Based on the equation $c = E(e, m)$, only Alice can decrypt c using her private key, with $m = D(d, c)$, where m is the message.

5. If anyone else wants to send a message to Alice, they can also use her public key **e**.

The system is secure from a decryption perspective because only Alice can decrypt the message, but Alice can never be sure if Bob sent the message, as the public key **e** can be used by anyone.

RSA

Rivest, Shamir, Adleman (1977) developed the RSA algorithm, which is based on exponentiation and modular arithmetic (remainder division). The RSA encryption relies on the fact that if the following equation holds:

$$T^d \mod N = T$$

Then the equation can be split into two parts, where the first equation represents encryption and the second represents decryption:

- **Encryption**: $C = T^e \mod N$
- **Decryption**: $T = C^d \mod N$

However, this relationship does not work for just any arbitrary choice of e, d, and N. The algorithm relies on specific conditions for these values. Let’s explore what conditions are required for the RSA algorithm to function correctly.

How RSA Works

The RSA algorithm uses both **public** and **private keys**. For it to work correctly, the following steps are followed:

1.) **Key Generation**:

- Choose two large prime numbers: p and q.
- Calculate $N = p \times q$.
- Compute Euler's totient function $\phi(N) = (p-1) \times (q-1)$. (totient function)
- Choose a public exponent e such that e is relatively prime to $\phi(N)$ (i.e., $1 < e < \phi(N)$ and $\gcd(e, \phi(N)) = 1$).
- Compute the private key d, which is the modular multiplicative inverse of e, meaning $d \times e \equiv 1 \mod \phi(N)$.

2.) **Encryption**:

The message T is encrypted using the public key: $C = T^e \mod N$.

3. **Decryption**:

The ciphertext C is decrypted using the private key: $T = C^d \mod N$.

Conditions for the Keys

The equations work correctly only when specific conditions are met:

- N must be large enough to prevent attackers from easily factoring N into p and q, which would compromise the encryption.
- The public exponent e and the private exponent d must have a specific relationship with N and Euler's totient $\phi(N)$. The relationship between e and d is that $e \times d \equiv 1 \mod \phi(N)$, ensuring that decryption reverses the encryption process.

Summary

The RSA algorithm depends on the correct choice of prime numbers p and q, the calculation of N, and the mathematical relationship between the exponents e (public key) and d (private key). Without these specific conditions, the encryption and decryption process will not work properly.

Example

Step 1: Key Generation

1.) **Choose two large prime numbers** p and q:

- $p = 61$
- $q = 53$

2.) **Compute \(N \)** (the modulus):

\[N = p \times q = 61 \times 53 = 3233 \] So, \(N = 3233 \).

3.) **Compute Euler’s totient function** \(\phi(N) \):

\[\phi(N) = (p - 1) \times (q - 1) = (61 - 1) \times (53 - 1) = 60 \times 52 = 3120 \] So, \(\phi(N) = 3120 \).

4. **Choose a public exponent** \(e \), such that \(1 < e < \phi(N) \) and \(\gcd(e, \phi(N)) = 1 \):

- Let’s choose \(e = 17 \), which is relatively prime to 3120.

5. **Calculate the private exponent** \(d \), which is the modular multiplicative inverse of \(e \mod \phi(N) \): \[d \times e \equiv 1 \mod \phi(N) \] Using the extended Euclidean algorithm, we find that \(d = 2753 \).

Step 2: Encryption

Now that the keys are set, let’s encrypt a message. Suppose our message is a number \(m = 65 \).

1.) **Public key** is \((e, N) = (17, 3233) \).

2.) **Encryption formula**: \[c = m^e \mod N \]

- \(m = 65 \), \(e = 17 \), \(N = 3233 \).
- Compute \(65^{17} \mod 3233 \):

Using modular exponentiation: \[65^{17} \mod 3233 = 2790 \]

So, the **ciphertext** \(c = 2790 \).

Step 3: Decryption

Now, Alice receives the ciphertext \(c = 2790 \) and uses her private key \(d = 2753 \) to decrypt the message.

1.) **Private key** is \((d, N) = (2753, 3233) \). 2.) **Decryption formula**: \[m = c^d \mod N \]

- \(c = 2790 \), \(d = 2753 \), \(N = 3233 \).
- Compute \(2790^{2753} \mod 3233 \):

Again, using modular exponentiation: \[2790^{2753} \mod 3233 = 65 \]

Thus, Alice successfully decrypts the message and retrieves the original plaintext **65**.

Summary of the RSA Example:

- Public key: \((e = 17, N = 3233) \)

- Private key: $(d = 2753, N = 3233)$
- Message to encrypt: $m = 65$
- Ciphertext: $c = 2790$
- Decrypted message: $m = 65$

This is a simple RSA example. In real-world applications, much larger prime numbers p and q are used to ensure security.